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# Notes on Re-nnd Generalized Inverses

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**Abstract.** Motivated by a recent paper, in which the authors studied Re-nnd $\{1,3\}$-inverse, $\{1,4\}$-inverse and $\{1,3,4\}$-inverse of a square matrix, in this paper, we establish some equivalent conditions for the existence of Re-nnd $\{1,2,3\}$-inverse, $\{1,2,4\}$-inverse and $\{1,3,4\}$-inverse. Furthermore, some expressions of these generalized inverses are presented.

## 1. Introduction

Let $\mathbb{C}^{m\times n}$ denote the set of all $m \times n$ matrices over the complex field $\mathbb{C}$, $\mathbb{C}_H^m$ denote the set of all $m \times m$ Hermitian matrices. For $A \in \mathbb{C}^{m\times n}$, its rank and conjugate transpose will be denoted by $r(A)$ and $A^*$ respectively. We write $A \geqslant 0$ (or $A > 0$) if $A$ is positive semidefinite matrix (or positive definite matrix). For Hermitian matrix $A$, its positive and negative index of inertia are symbolled by $i_+(A)$ and $i_-(A)$ respectively.

For a matrix $A \in \mathbb{C}^{m\times n}$, the Moore-Penrose inverse $A^\dagger$ is defined to be the unique solution of the four Penrose equations [1]

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA.$$

Let $\emptyset \neq \eta \subseteq \{1,2,3,4\}$. Then $A\eta$ denotes the set of all matrices $X$ satisfy $(i)$ for all $i \in \eta$. Any matrix $X \in A\eta$ is called an $\eta$-inverse of $A$. One usually denotes any $\{1,2,3\}$-inverse of $A$ as $A^{(1,2,3)}$, any $\{1,2,4\}$-inverse of $A$ is denoted by $A^{(1,2,4)}$, and any $\{1,3,4\}$-inverse of $A$ is denoted by $A^{(1,3,4)}$. Let $A_{re}^{(i,j,\cdots,k)}$ be the Re-nnd $\{i,j,\cdots,k\}$-inverse of $A$. For convenience, we denote $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$.

For a matrix $A \in \mathbb{C}^{n\times n}$, the group inverse, denoted by $A^\#$, is the unique matrix $X$ satisfying

$$AXA = A,\quad XAX = X,\quad AX = XA.$$

Recently, some authors studied several special generalized inverses, such as Hermitian generalized inverses, positive semidefinite generalized inverses and Re-nnd generalized inverses of a square matrix. For example, Tian [2] presented a general expression for each Hermitian generalized inverse of a Hermitian matrix; Liu and Yang [3] investigated Hermitian $\{1,3\}$-inverse and $\{1,4\}$-inverse; newly, some in-depth

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researches are done on Hermitian generalized inverses and positive semidefinite generalized inverses by Liu [4]; Nikolov and Cvetković-Ilić [5] studied Re-nnd $\{1,3\}$-inverse, $\{1,4\}$-inverse and $\{1,3,4\}$-inverse, also positive semidefinite $\{1,3\}$-inverse and $\{1,4\}$-inverse. For Re-nnd $\{1\}$-inverse, it can be regarded as the Re-nnd solution to equation $AXA = A$, which has been considered by Cvetković-Ilić [6].

Motivated by the above work, in this article, we establish some conditions for the existences of Re-nnd $\{1,2,3\}$-inverse, $\{1,2,4\}$-inverse and $\{1,3,4\}$-inverse, moreover, expressions of these Re-nnd generalized inverses are given.

Before giving the main results, we first introduce several lemmas as follows.

**Lemma 1.1.** [7] *Let $A \in \mathbb{C}^m_H$, $B \in \mathbb{C}^{m\times n}$ and $C \in \mathbb{C}^{p\times m}$ be given. Then*

$$\min_{X\in\mathbb{C}^{n\times m}} i_\pm[A - BXC - (BXC)^*] = r\begin{pmatrix} A & B & C^* \end{pmatrix} + \max\{i_\pm(M_1) - r(N_1),\ \ i_\pm(M_2) - r(N_2)\},$$

*where*

$$M_1 = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix},\ M_2 = \begin{pmatrix} A & C^* \\ C & 0 \end{pmatrix},\ N_1 = \begin{pmatrix} A & B & C^* \\ B^* & 0 & 0 \end{pmatrix},\ N_2 = \begin{pmatrix} A & B & C^* \\ C & 0 & 0 \end{pmatrix}.$$

**Lemma 1.2.** [7] *Let $A \in \mathbb{C}^m_H$, $B \in \mathbb{C}^{m\times n}$, and denote $M = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix}$. Then*

$$i_\pm(M) = r(B) + i_\pm(E_B A E_B).$$

**Lemma 1.3.** [8] *Let $A \in \mathbb{C}^{m\times n}$. Then*

$$\begin{aligned} A^{(1,2,3)} &= A^\dagger + F_A V_1 A A^\dagger, \\ A^{(1,2,4)} &= A^\dagger + A^\dagger A V_2 E_A, \\ A^{(1,3,4)} &= A^\dagger + F_A V_3 E_A, \end{aligned}$$

*where $V_i$ $(i = 1,2,3)$ are arbitrary matrices with proper sizes.*

**Lemma 1.4.** [9] *Let $A \in \mathbb{C}^{m\times p}$ and $B \in \mathbb{C}^{q\times m}$ be given, and define $M = \begin{pmatrix} E_A & F_B \end{pmatrix}$, $G = \begin{pmatrix} A & B^* \end{pmatrix}$ and $H = \begin{pmatrix} B^* & A \end{pmatrix}^*$. Then the general solution of $AXB + (AXB)^* \geqslant 0$ can be written in the parametric form*

$$X = A^\dagger E_M U U^* E_M B^\dagger + \begin{pmatrix} I_p & 0 \end{pmatrix} F_G W E_H \begin{pmatrix} I_q \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & I_p \end{pmatrix} E_H W^* F_G \begin{pmatrix} 0 \\ I_q \end{pmatrix} + F_A W_1 + W_2 E_B,$$

*where $U \in \mathbb{C}^{m\times m}$, $W \in \mathbb{C}^{(p+q)\times(p+q)}$ and $W_1, W_2 \in \mathbb{C}^{p\times q}$ are arbitrary.*

**Lemma 1.5.** [8] *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then*

$$r\begin{pmatrix} A & B \end{pmatrix} = r(A) + r(E_A B),$$
$$r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A).$$

**Lemma 1.6.** [5] *Let $A \in \mathbb{C}^{n\times n}$. Then $A^{(1,3)}_{re}$ exists if and only if $(A^\dagger)^2 A$ (or $A^2 A^\dagger, A^* A^2$) is Re-nnd; $A^{(1,4)}_{re}$ exists if and only if $A(A^\dagger)^2$ (or $A^\dagger A^2, A^2 A^*$) is Re-nnd.*

**Lemma 1.7.** [10] *Let $A, C \in \mathbb{C}^{n\times m}$, and $B, D \in \mathbb{C}^{m\times n}$, such that both $AX = C$ and $XB = D$ have a Re-nnd solution. If the pair of equations have a common solution (i.e. $AD = CB$), then there exists a common Re-nnd solution if and only if*

$$r\begin{pmatrix} A & C \\ B^* & -D^* \end{pmatrix} = r\begin{pmatrix} A & CA^* \\ B^* & -D^*A^* \end{pmatrix} = r\begin{pmatrix} A & CB \\ B^* & -D^*B \end{pmatrix}.$$

## 2. Re-nnd Generalized Inverses

In this section, our purpose is to present some conditions for Re-nnd {1, 2, 3}-inverse, {1, 2, 4}-inverse and {1, 3, 4}-inverse existing, and then establish several expressions for these Re-nnd generalized inverses.

**Theorem 2.1.** *Let $A \in \mathbb{C}^{m\times m}$. Then the following statements are equivalent:*
*(i) $A_{re}^{(1,2,3)}$ exists;*
*(ii) $(A^{\dagger})^2A$ is Re-nnd and $r(A) = r(A^2)$;*
*(iii) $A^2A^{\dagger}$ is Re-nnd and $r(A) = r(A^2)$;*
*(iv) $A^*A^2$ is Re-nnd and $r(A) = r(A^2)$;*
*(v) $A^{\#}AA^{\dagger}$ is Re-nnd and $r(A) = r(A^2)$.*
*In this case, then*

$$X = A^{\#}AA^{\dagger} + \begin{pmatrix} F_A & 0 \end{pmatrix} F_G W E_H \begin{pmatrix} AA^{\dagger} \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & F_A \end{pmatrix} E_H W^* F_G \begin{pmatrix} 0 \\ AA^{\dagger} \end{pmatrix} \tag{1}$$

*is a Re-nnd {1, 2, 3}-inverse of A, where $G = \begin{pmatrix} F_A & AA^{\dagger} \end{pmatrix}$, $H = \begin{pmatrix} AA^{\dagger} & F_A \end{pmatrix}^*$, and $W \in \mathbb{C}^{2m\times 2m}$ is arbitrary.*

*Proof.* Since $A^{(1,2,3)} = A^{\dagger} + F_AVAA^{\dagger}$, then $A_{re}^{(1,2,3)}$ exists if and only if there exists some $V$ such that $A^{(1,2,3)}$ is Re-nnd, i.e.,

$$\min_V i_-\left(A^{(1,2,3)} + (A^{(1,2,3)})^*\right) = \min_V i_-\left(A^{\dagger} + (A^{\dagger})^* + F_AVAA^{\dagger} + (F_AVAA^{\dagger})^*\right) = 0.$$

By Lemma 1.1, we have

$$\begin{aligned}
& \min_V i_-\left(A^{\dagger} + (A^{\dagger})^* + F_AVAA^{\dagger} + (F_AVAA^{\dagger})^*\right) \\
= \ & \min_V i_-\left(A^{\dagger} + (A^{\dagger})^* - (-F_AVAA^{\dagger}) - (-F_AVAA^{\dagger})^*\right) \\
= \ & r\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A & AA^{\dagger} \end{pmatrix} + \max\left\{ i_-\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A \\ -F_A & 0 \end{pmatrix} - r\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A & AA^{\dagger} \\ -F_A & 0 & 0 \end{pmatrix}, \right. \\
& \left. i_-\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & AA^{\dagger} \\ AA^{\dagger} & 0 \end{pmatrix} - r\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A & AA^{\dagger} \\ AA^{\dagger} & 0 & 0 \end{pmatrix} \right\}.
\end{aligned}$$

On account of Lemma 1.2 and Lemma 1.5, we get

$$\begin{aligned}
r\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A & AA^{\dagger} \end{pmatrix} &= r\begin{pmatrix} A^{\dagger} & I_m & AA^{\dagger} \end{pmatrix} = m, \\
i_-\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A \\ -F_A & 0 \end{pmatrix} &= r(-F_A) + i_-[A^{\dagger}A(A^{\dagger} + (A^{\dagger})^*)A^{\dagger}A] \\
&= r(F_A) + i_-[(A^{\dagger})^2A + ((A^{\dagger})^2A)^*], \\
i_-\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & AA^{\dagger} \\ AA^{\dagger} & 0 \end{pmatrix} &= r(AA^{\dagger}) + i_-[E_A(A^{\dagger} + (A^{\dagger})^*)E_A] = r(A), \\
r\begin{pmatrix} A^{\dagger} + (A^{\dagger})^* & -F_A & AA^{\dagger} \\ -F_A & 0 & 0 \end{pmatrix} &= r\begin{pmatrix} A^{\dagger} & F_A & AA^{\dagger} \\ F_A & 0 & 0 \end{pmatrix} \\
&= r(F_A) + r\begin{pmatrix} (A^{\dagger})^2A & F_A & AA^{\dagger} \end{pmatrix} \\
&= 2r(F_A) + r\begin{pmatrix} (A^{\dagger})^2A & A^{\dagger}A^2A^{\dagger} \end{pmatrix} \\
&= 2r(F_A) + r\begin{pmatrix} A(A^{\dagger})^2A & A^2A^{\dagger} \end{pmatrix} \\
&= 2r(F_A) + r\begin{pmatrix} A(A^{\dagger})^2A & A^2 \end{pmatrix},
\end{aligned}$$

$$r\begin{pmatrix} A^{\dagger}+(A^{\dagger})^{*} & -F_A & AA^{\dagger} \\ AA^{\dagger} & 0 & 0 \end{pmatrix} = r\begin{pmatrix} 0 & F_A & AA^{\dagger} \\ AA^{\dagger} & 0 & 0 \end{pmatrix}$$
$$= r(AA^{\dagger}) + r(F_A) + r(A^{\dagger}A^2A^{\dagger})$$
$$= m + r(A^2).$$

Hence,

$$\begin{aligned} & \min_V i_-\left(A^{\dagger}+(A^{\dagger})^{*}+F_AVAA^{\dagger}+(F_AVAA^{\dagger})^{*}\right) \\ = \ & m+\max\left\{i_-[(A^{\dagger})^2A+((A^{\dagger})^2A)^{*}]-r(F_A)-r\begin{pmatrix} A(A^{\dagger})^2A & A^2 \end{pmatrix},\ r(A)-m-r(A^2)\right\} \\ = \ & \max\left\{i_-[(A^{\dagger})^2A+((A^{\dagger})^2A)^{*}]+r(A)-r\begin{pmatrix} A(A^{\dagger})^2A & A^2 \end{pmatrix},\ r(A)-r(A^2)\right\}. \end{aligned} \tag{2}$$

Letting the right hand side of (2) be zero produces

$$i_-[(A^{\dagger})^2A+((A^{\dagger})^2A)^{*}]=0,\ r(A)=r\begin{pmatrix} A(A^{\dagger})^2A & A^2 \end{pmatrix},\ r(A)=r(A^2)$$

which are equivalent to $(A^{\dagger})^2A$ is Re-nnd and $r(A)=r(A^2)$. So (*i*) and (*ii*) are equivalent. And the equivalence of (*ii*), (*iii*) and (*iv*) are followed by Theorem 2.1 in [5].

Next, we show that (*iii*) and (*v*) are also equivalent. If $A^2A^{\dagger}$ is Re-nnd and $r(A)=r(A^2)$, we can deduce

$$\begin{aligned} & A^2A^{\dagger}+(A^2A^{\dagger})^{*}\geqslant 0 \\ \Rightarrow \ & A^{\#}\left(A^2A^{\dagger}+(A^2A^{\dagger})^{*}\right)(A^{\#})^{*}\geqslant 0 \\ \Rightarrow \ & A^{\#}AA^{\dagger}+(A^{\#}AA^{\dagger})^{*}\geqslant 0, \end{aligned}$$

which means that $A^{\#}AA^{\dagger}$ is Re-nnd.
Similarly, we can prove (*v*) $\Rightarrow$ (*iii*).

If $A_{re}^{(1,2,3)}$ exists, suppose $X=A^{\#}AA^{\dagger}+F_AVAA^{\dagger}$. It is easy to verify that $X$ is a $\{1,2,3\}$-inverse of $A$. Although it is very difficult to give a general expression of $V$ such that $A^{\#}AA^{\dagger}+F_AVAA^{\dagger}$ is Re-nnd, specially, we can choose some $V$ satisfying $F_AVAA^{\dagger}$ is Re-nnd, i.e.,

$$F_AVAA^{\dagger}+(F_AVAA^{\dagger})^{*}\geqslant 0. \tag{3}$$

In view of Lemma 1.4, the general solution of (3) can be written in the parametric form

$$V=F_AE_MUU^{*}E_MAA^{\dagger}+\begin{pmatrix} I_m & 0 \end{pmatrix}F_GWE_H\begin{pmatrix} I_m \\ 0 \end{pmatrix}-\begin{pmatrix} 0 & I_m \end{pmatrix}E_HW^{*}F_G\begin{pmatrix} 0 \\ I_m \end{pmatrix}+A^{\dagger}AW_1+W_2E_A,$$

where $M=\begin{pmatrix} A^{\dagger}A & E_A \end{pmatrix}$, $G=\begin{pmatrix} F_A & AA^{\dagger} \end{pmatrix}$, $H=\begin{pmatrix} AA^{\dagger} & F_A \end{pmatrix}^{*}$, and $U, W, W_1, W_2$ are arbitrary.
In addition, it follows from Lemma 1.5 and $r(A^2)=r(A)$ that

$$r(M)=r\begin{pmatrix} A^{\dagger}A & E_A \end{pmatrix}=r\begin{pmatrix} A^{*} & F_{A^{*}} \end{pmatrix}=r\begin{pmatrix} A^{*} & I_m \\ 0 & A^{*} \end{pmatrix}-r(A)=m,$$

which means that $E_M=0$.

So, (1) can be obtained immediately. □

In an analogous way, the following result can be deduced.

**Theorem 2.2.** *Let $A\in\mathbb{C}^{m\times m}$. Then the following statements are equivalent:*
(*i*) $A_{re}^{(1,2,4)}$ *exists;*
(*ii*) $A(A^{\dagger})^2$ *is Re-nnd and* $r(A)=r(A^2)$*;*
(*iii*) $A^{\dagger}A^2$ *is Re-nnd and* $r(A)=r(A^2)$*;*
(*iv*) $A^2A^{*}$ *is Re-nnd and* $r(A)=r(A^2)$*;*

*(v)* $A^{\dagger}AA^{\#}$ *is Re-nnd and* $r(A) = r(A^2)$.
*In this case, then*

$$X = A^{\dagger}AA^{\#} + \begin{pmatrix} A^{\dagger}A & 0 \end{pmatrix} F_G W E_H \begin{pmatrix} E_A \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & A^{\dagger}A \end{pmatrix} E_H W^* F_G \begin{pmatrix} 0 \\ E_A \end{pmatrix}$$

*is a Re-nnd* $\{1,2,4\}$*-inverse of* $A$, *where* $G = \begin{pmatrix} A^{\dagger}A & E_A \end{pmatrix}$, $H = \begin{pmatrix} E_A & A^{\dagger}A \end{pmatrix}^*$, *and* $W \in \mathbb{C}^{2m\times 2m}$ *is arbitrary.*

In [5], the authors presented some conditions for the existence for $A_{re}^{(1,3,4)}$, next, we give some new conditions.

**Theorem 2.3.** *Let* $A \in \mathbb{C}^{m\times m}$. *Then the following statements are equivalent:*
*(i)* $A_{re}^{(1,3,4)}$ *exists;*
*(ii)* $A_{re}^{(1,3)}$, $A_{re}^{(1,4)}$ *exist, and*

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r(A) + r(A^*A + A^2) = r(A) + r(AA^* + A^2). \tag{4}$$

*Proof.* Since the $A_{re}^{(1,3,4)}$ can be regarded as the common Re-nnd solution to $A^*AX = A^*$ and $XAA^* = A^*$. By Lemma 1.7, we get that statement *(i)* is equivalent to $A_{re}^{(1,3)}$, $A_{re}^{(1,4)}$ exist, and

$$r\begin{pmatrix} A^*A & A^* \\ AA^* & -A \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2A \\ AA^* & -AA^*A \end{pmatrix} = r\begin{pmatrix} A^*A & A^*AA^* \\ AA^* & -A^2A^* \end{pmatrix}.$$

Moreover,

$$\begin{aligned} r\begin{pmatrix} A^*A & (A^*)^2A \\ AA^* & -AA^*A \end{pmatrix} &= r\begin{pmatrix} A^*A & (A^*)^2 \\ AA^* & -AA^* \end{pmatrix} = r\begin{pmatrix} A^*A & (A^*)^2 \\ A^* & -A^* \end{pmatrix} \\ &= r(A^*) + r[A^*A + (A^*)^2] = r(A) + r(A^*A + A^2), \\ r\begin{pmatrix} A^*A & A^*AA^* \\ AA^* & -A^2A^* \end{pmatrix} &= r\begin{pmatrix} A^*A & A^*A \\ AA^* & -A^2 \end{pmatrix} = r\begin{pmatrix} A & A \\ AA^* & -A^2 \end{pmatrix} \\ &= r(A) + r(AA^* + A^2). \end{aligned}$$

According to the above analyses, (4) is valid. The proof is complete. □

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